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SOME NEW KRULL-SCHIMIDT CATEGORIES WITH AN APPLICATION

Du Jiachun*, Huang Hualin and Ye Yu***

** Department of Mathematics, University of Science and Technology of China, Hefei 230026, China.* **E-mail:** *jcdu@mail.ustc.edu.cn; yeyu@ustc.edu.cn*
*** School of Mathematics and System Sciences, Shandong University, Jinan 250100, China.* **E-mail:** *hualin@sdu.edu.cn*
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By using Warfield's Lemma, three new Krull-Schmidt categories are found. As an application, the unicity of gradings on certain (co)modules is obtained.

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§1. INTRODUCTION

The classical Krull-Schmidt theorem, which was discovered in the 1920s, also known as the Krull-Remak-Schmidt theorem, states that any two direct sum decompositions of a module (over a ring) of finite length into indecomposable summands are isomorphic. It is one of the most basic results in module (or representation) theory.

Nowadays the unique decomposition property (for general mathematical objects) is always called Krull-Schmidt property. An additive category $\mathcal{C}$ is said to be a Krull-Schmidt category (see [1, p. 52]) provided that the endomorphism ring $\mathrm{End}(M)$ of any indecomposable object M of $\mathcal{C}$ is a local ring. Such category assures the Krull-Schmidt property for each object. Therefore in a Krull-Schmidt category, the classification problem for objects reduces to that for indecomposable ones. Naturally it is always very important, and tempting mathematicians to find the Krull-Schmidt property for a category.

The aim of this short note is to present the Krull-Schmidt property for three kinds of categories. As an application, the unicity of gradings on the indecomposables in certain categories is obtained.

§2. THREE KINDS OF NEW KRULL-SCHMIDT CATEGORIES

We begin by introducing three kinds of categories.

(1) Let R be an arbitrary ring with unit. Denote the category of left R-modules by R-Mod. A module M is said to be locally-finite provided that every cyclic submodule is of finite length. A semi-simple module M is said to be multiplicity-bounded if there exists a positive integer n such that the multiplicity of any simple appearing in the decomposition of M is bounded by n. Consider the following two full subcategories of R-Mod

$$F(R) := \{M \mid M \text{ is locally-finite with multiplicity-bounded socle}\},$$
$$FF(R) := \{M \mid M \text{ is locally-finite with finite length socle}\}.$$

Clearly, $FF(R) \subseteq F(R)$.

(2) Let k be a commutative artinian ring, A an arbitrary k-algebra. By A-Mod we denote the category of left A-modules. An A-module M is said to be locally-finite over k if every cyclic submodule is of finite length over k. Consider the following two full subcategories of A-Mod

$$F(A;k) := \{M \mid M \text{ is locally-finite over } k \text{ with multiplicity-bounded socle}\},$$
$$FF(A;k) := \{M \mid M \text{ is locally-finite over } k \text{ with socle being of finite length over } k\}.$$

Clearly $FF(A;k) \subseteq F(A;k)$.

(3) Let k be any field, C an arbitrary k-coalgebra (see [2]). Denote by $\mathcal{M}^C$ the category of right C-comodules. Consider the following two full subcategories of $\mathcal{M}^C$

$$F(C) := \{M \mid M \text{ has multiplicity-bounded socle}\},$$
$$FF(C) := \{M \mid M \text{ has finite-dimensional socle}\}.$$

Clearly $FF(C) \subseteq F(C)$.

Our main observation is

Theorem 2.1. The following three categories are Krull-Schmidt: (1) $FF(R)$ for any commutative ring R; (2) $FF(A;k)$ for any algebra A over an arbitrary commutative artinian ring k; (3) $FF(C)$ for any coalgebra C over an arbitrary field k.

We would like to leave two comments on Theorem 2.1: (i) Let R be a commutative ring, R-Art the category of artinian modules. Then it is a good exercise to show that $R\text{-Art} \subseteq FF(R)$. Thus from the case (1) in Theorem 2.1 one deduces that R-Art is a Krull-Schmidt category, which has been known in [3]. (ii) One could ask whether the category $FF(R)$ is Krull-Schimidt for non-commutative noetherian or local rings and what kind of full subcategories of R-Mod are Krull-Schmidt for general R. As a matter of fact, these problems are far from known. The readers are referred to [4–6] for related subjects, in particular results about the category of artinian modules over local rings.

The proof of Theorem 2.1 relies heavily on the following result of Warfield (see [3]). We refer it as Warfield's Lemma, which generalizes the classical Fitting's Lemma.

Lemma 2.1 (Warfield's Lemma). Let R be a ring with unit, M an R-module, $f : M \longrightarrow M$ an endomorphism. Assume that the following condition is fulfilled:

(CWL) there is a family of submodules M_i, $i \in \Lambda$, such that each M_i is of finite length, $f(M_i) \subseteq M_i$ and $M = \sum_{i\in\Lambda} M_i$. Then there exists a decomposition of modules

$$M = N \oplus H$$

such that

(1) $f(N) \subseteq N$ and the restriction $f|_N : N \longrightarrow N$ is locally-nilpotent;

(2) $f(H) \subseteq H$ and the restriction $f|_H : H \longrightarrow H$ is an isomorphism.

Consequently, we have

(a) f is injective if and only if f is an isomorphism.

(b) if in addition M is indecomposable, then f or $\mathrm{Id}_M - f$ is invertible.

Proof. Set

$$N = \sum_{n\geq 1} \mathrm{Ker} f^n \quad \text{and} \quad H = \sum_{i\in\Lambda} \Big(\bigcap_{n\geq 1} f^n(M_i) \Big).$$

Applying the classical Fitting's Lemma to $f|_{M_i} : M_i \longrightarrow M_i$, we get

$$M_i = (N \cap M_i) \oplus \Big(\bigcap_{n\geq 1} f^n(M_i) \Big)$$

and the restriction of f to $\bigcap_{n\geq 1} f^n(M_i)$ is an isomorphism, $i \in \Lambda$. Then it is not hard to see that the result follows.

For the consequence (a), just note that $N = 0$; for (b), if f is not an isomorphism, then f is locally-nilpotent, and thus $\mathrm{Id}_M - f$ has the inverse given by $\sum_{n\geq 0} f^n$ (which is well-defined by the local-nilpotency).

We have the following useful corollary:

Corollary 2.1. (1) Let R be a commutative ring. Then for any indecomposable $M \in F(R)$, its endomorphism ring $\mathrm{End}_R(M)$ is local. (2) Let k be a commutative artinian ring, A a k-algebra. Then for any indecomposable $M \in F(A; k)$, its endomorphism ring $\mathrm{End}_A(M)$ is local.

Proof. (1) In view of Consequence (b) in Warfield's Lemma, all we need to do is to verify the condition (CWL). For each $m \in M$, consider the submodule M_m generated by $\{f^n(m) \mid n \geq 0\}$. Note that

$$f(M_m) \subseteq M_m \quad \text{and} \quad M = \sum_{m \in M} M_m.$$

We need only to show that each M_m is of finite length.

In fact, we can take the annihilator ideal $I = \mathrm{Ann}(Rm)$ and the factor ring $\bar{R} = R/I$. Thus we can view Rm and M_m as $\bar{R}$-modules. Since Rm is of finite length and it is a faithful $\bar{R}$-module, the map $i : \bar{R} \longrightarrow M_m,\ r \mapsto rm$ makes $\bar{R}$ a submodule of M_m. Note that we have used the commutativity of R. From this it follows that $\bar{R}$ is a commutative artinian ring. Since the socle of M is multiplicity-bounded, so is the socle of M_m. On the other hand, the ring $\bar{R}$ has only finitely many simples, thus the socle $\mathrm{soc}(M_m)$ of M_m is of finite length. Note that $\mathrm{soc}(M_m) \subseteq M_m$ is essential. Thus we have an embedding $M_m \subseteq E(\mathrm{soc}(M_m))$, where $E(\mathrm{soc}(M_m))$ denotes the injective hull in the category of $\bar{R}$-modules. Since $E(\mathrm{soc}(M_m))$ is of finite length, so is M_m. This completes the proof.

(2) Similarly we also need to verify the condition (CWL) of Lemma 2.1. Let $M \in F(A; k)$. For each $m \in M$, consider the same M_m as above. We claim that M_m is of finite length. In fact, it is of finite length even over k.

Similar to (1), take $I = \mathrm{Ann}(Am)$ and $\bar{A} = A/I$. Then it is not hard to see that $\bar{A}$ is an artinian algebra over k. Now applying the same argument as above, one can show the claim.

Proof of Theorem 2.1. Considering the length of the socles, we know that each object in the three categories can be written as a finite direct sum of indecomposables. Now the Krull-Schmidt properties of (1) and (2) follow immediately from Corollary 2.1.

To see (3), recall that the right C-comodules can be regarded as the left rational C^*-modules, where C^* is the dual algebra. So we have a natural full embedding $\Phi : \mathcal{M}^C \longrightarrow C^*$-Mod. Note that $\Phi(\mathrm{soc}(M)) = \mathrm{soc}(\Phi(M))$. Thus Φ identifies $FF(C)$ (resp. $F(C)$) as a full subcategory $FF(C^*)$ (resp. $F(C^*)$), which is closed under direct summands. Thus the Krull-Schmidt property of (3) follows from (2).

§3. AN APPLICATION: UNICITY OF GRADINGS

We will give an application of the obtained results to the unicity of gradings on certain modules (see [7, 8]).

Let G be an arbitrary group. We write it multiplicatively and denote its unit by e. Let k be a commutative artinian ring, $A = \bigoplus_{g \in G} A_g$ be a G-graded k-algebra. We consider the category A-Gr of left graded A-modules and the following grading-forgetful functor

$$U : A\text{-Gr} \longrightarrow A\text{-Mod}.$$

A-modules lying in the essential image of U are called gradable modules.

Recall the degree-shift functors on A-Gr. Graded modules will be written as $M = \bigoplus_{g \in G} M_g$ satisfying $A_{g'} M_g \subseteq M_{g'g}$. For each $h \in G$, we define a new graded module $M(h)$ such that

$M(h)_g = M_{hg}$, $g \in G$. Thus we define the degree-shift endofunctor (h) on A-Gr. Observe that $U \circ (h) = U$ for all $h \in G$. An A-module X is called uniquely-gradable if X is gradable and for any two graded modules M, N with $U(M) \simeq U(N) \simeq X$ there exists $h \in G$ such that $M \simeq N(h)$. Let us remark that the question of whether a module is gradable or uniquely-gradable is quite subtle in representation theory (see [9]).

Taking advantage of the observation in the previous section, we get the main result of this section, which claims that in some certain Krull-Schmidt categories the indecomposable objects are uniquely-gradable.

Theorem 3.1. Let k be a commutative artinian ring, $A = \bigoplus_{g\in G} A_g$ be a G-graded k-algebra. Let $X \in FF(A;k)$ be indecomposable. Assume that X is gradable. Then X is uniquely-gradable.

Proof. Suppose that $M = \bigoplus_{g\in G} M_g$ and $N = \bigoplus_{g\in G} N_g$ are two graded A-modules such that $U(M) \simeq U(N) \simeq X$. We need to show that $M \simeq N(h)$ for some $h \in G$.

Assume that $f : U(M) \longrightarrow U(N)$ is an isomorphism. For each $h \in G$, consider $f_h : M \longrightarrow N(h)$ such that for each g, the map $f_h|_{M_g} : M_g \longrightarrow N_{hg}$ is given by $\pi_{hg} \circ f|_{M_g}$, where $\pi_{hg} : U(N) \longrightarrow N_{hg}$ is the canonical projection. It is direct to check that each f_h is a morphism in A-Gr, and for each $m \in U(M)$, only finitely many $f_h(m)$ are non-zero and $f(m) = \sum_{h\in G} f_h(m)$.

We claim that there are only finitely many $h \in G$ such that $U(f_h)$ does not vanish on $\mathrm{soc}(U(M))$. If so, write the collection of these h's by Λ. Consider $f' := \sum_{h\in\Lambda} U(f_h) : U(M) \longrightarrow U(N)$. Therefore the restrictions of f' and f to $\mathrm{soc}(U(M))$ are the same. In particular, $f'|_{\mathrm{soc}(U(M))}$ is injective. Since $\mathrm{soc}(U(M)) \subseteq U(M)$ is essential, f' is injective. Note that $U(M) \simeq U(N) \simeq X$. Thus by (a) in Warfield's Lemma, f' is an isomorphism. By Theorem 2.1, $\mathrm{End}_A(X)$ is local. Hence $f' = \sum_{h\in\Lambda} U(f_h)$ is an isomorphism, which implies that there exists some $h \in \Lambda$ such that $U(f_h)$ is an isomorphism. Therefore $f_h : M \longrightarrow N(h)$ is an isomorphism in A-Gr. Thus with the claim the theorem follows.

To finish the proof, we need to show the claim. Consider the following set

$$I_M := \{h \in G \mid \pi_h(\mathrm{soc}(U(M))) \neq 0\},$$

where $\pi_h : U(M) \longrightarrow M_h$ is the canonical projection. Since $\mathrm{soc}(U(M)) \simeq \mathrm{soc}(X)$ is of finite length over k, I_M is a finite set. Similarly we have the finite set I_N. Consider $f_h : M \longrightarrow N(h)$, and note that $U(f_h)$ sends socle to socle. Consider certain homogeneous components. Since $U(f_h)$ does not vanish on $\mathrm{soc}(U(M))$, h necessarily lies in the following set

$$\{h \in G \mid I_M h \cap I_N \neq \varnothing\},$$

which is certainly a finite set. Thus we have the claim.

Using similar argument, we have the following result.

Theorem 3.2. Let k be a field, $C = \bigoplus_{C\in G} C_g$ be a G-graded k-coalgebra. Let $X \in FF(C)$ be indecomposable. Assume that X is gradable. Then X is uniquely-gradable.

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